Graduate Labor Economics

Sorting in the Labor Market (in Theory)

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Table of contents

Two-Sided Matching

How does matching differ from standard markets?

1. There is no price signal (no walrasian auctioneer)
2. Preferences are over agents not over goods.
3. There are indivisibilities. (Cannot match 30% with person A and 70% with person B. in general.)

Two-Sided Matching: Applications

- Online Dating
- Market design: doctor assignment to hospitals
- Kidney Exchange (google Al Roth Kidney Exchange)
- School Choice: Boston, New York (soon? SciencesPo)
- Gale and Shapley (1962)
    - pose problem
    - provide algorithm
    - show existence

One-to-One Matching: A Marriage Market

- Take two disjoint sets $W = \{w_1, \ldots, w_p\}$ and $M = \{m_1, \ldots, m_n\}$
- We want to match in paris (w_i, m_j) and allow for singles.
- Agents have preferences over members of other sex.
- This is just an ordered list:

$$P(m) = w_1, w_3, [m, w_p], \ldots, w_2$$

and similar for women.

One-to-One Matching

- We denote

$$\mathbf{P} = \{P(m_1), \ldots, P(m_n), P(w_1), \ldots, P(w_p)\}$$

  as the preference profile.

- The **marriage market** is defined by $(W, M, \mathbf{P})$

A particular men-to-women allocation is called a $\boxed{\text{matching}}$ μ:

Definition: Marriage Matching

A marriage matching μ is a one to one correspondence from $W \cup M$ onto itself, i.e. $\mu(\mu(x)) = x$, such that if $\mu(m) \neq m$ then $\mu(m) \in W$ and if $\mu(w) \neq w$ then $\mu(w) \in M$.

One-to-One Matching: Blocking μ

- a matching μ is **blocked by individual k** if k prefers being single to being matched with $\mu(k)$
- We write $k \succ_k \mu(k)$.
- A matching μ is **individually rational** if each agent in μ is *acceptable*, i.e. μ is not **blocked** by any agent.
- A matching μ is blocked by a **pair of agents** (m, w) if

$$w \succ_m \mu(m) \text{ and } m \succ_w \mu(w)$$

One-to-One Matching: Stable Matching

Definition: Stable Matching

A marriage matching μ **stable** if it is not blocked by any individual or any pair of agents.

Theorem: Gale and Shapley (1962)

A stable matching exists for every marriage market.

One-to-One Matching: Proof

- Their proof uses the **Deferred Acceptance Algorithm** (DAA).
- Start with one side of the market (men, say):

  Iter 1
    i. Each man proposes to his first choice (if any acceptable ones)
    ii. Each women holds their most preferred proposer

  Iter K ...

  Iter K+L STOP if no further proposals are made and match any woman to the man whose proposal she is currently holding.

- Break ties arbitrarily
- With finite set of men and women, this algo is finite and always stops.

One-to-One Matching: Proof

- Gives rise to a stable matching.
- Suppose not. Suppose m can do better, i.e. m prefers w to current match $\mu(m)$:
    1. $w \succ_m \mu(m)$
    2. m must have proposed to w before proposing to $\mu(m)$
    3. m must have been **rejected** by w
    4. that means that $\mu(w) \succ_w m$
    5. **Not** a blocking pair.
    6. Match is stable.

DAA Example

- Example: Consider market $(W, M, \mathbf{P})$ where

$$\begin{array}{ll} P(m_1) = w_1, w_2, w_3, w_4 & P(w_1) = m_2, m_3, m_1, m_4, m_5 \\ P(m_2) = w_4, w_2, w_3, w_1 & P(w_2) = m_3, m_1, m_2, m_4, m_5 \\ P(m_3) = w_4, w_3, w_1, w_2 & P(w_3) = m_5, m_4, m_1, m_2, m_3 \\ P(m_4) = w_1, w_4, w_3, w_2 & P(w_4) = m_1, m_4, m_5, m_2, m_3 \\ P(m_5) = w_1, w_2, w_4, m_5 & \end{array}$$

- The DAA proceeds as follows:

Iterate	w_1	w_2	w_3	w_4	(m_i)
1.	m_1, m_4, m_5			m_2, m_3	

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DAA Example

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1.	m_1, m_4, m_5			m_2, m_3	
2.	m_1	m_5	m_3	m_4, m_2	
3.	m_1	m_2, m_5	m_3	m_4	

DAA Example

- Example: Consider market $(W, M, \mathbf{P})$ where

$$
\begin{aligned}
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3.	m_1	m_2, m_5	m_3	m_4	
4.	m_1	m_2	m_3	m_4	m_5

DAA Example - M stable matching

- Example: Consider market $(W, M, \mathbf{P})$ where

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- Hence, the M-stable matching is:

$$\mu_M = \begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & (m_5) \\ m_1 & m_2 & m_3 & m_4 & (m_5) \end{array}$$

DAA Example - W stable matching

- Notice that if women were to make proposals, we'd get
- Hence, the stable matching is:

$$\mu_W = \begin{matrix} w_1 & w_2 & w_3 & w_4 & (m_5) \\ m_2 & m_3 & m_4 & m_1 & (m_5) \end{matrix}$$

- Implications:
  1. In general, the set of stable matchings is not a singleton.
  2. All m weakly prefer μ_M, opposite for women.
  3. I.e. there is a conflict between both sides of the market as to who is to make the offer!

One-to-one Matching Gale and Shapley

Theorem (Gale and Shapley)

When all men and women have strict preferences, there always exists an M-optimal stable matching, and a W-optimal stable matching. Furthermore, the matching μ_M produced by the DAA with men proposing is the M-optimal stable matching. The W-optimal stable matching is the matching μ_W produced by the DAA when women propose.

DDA in practice

look at the example!

Frictionless Matching

Introduction

Non-Transferrable Utility

Transferrable Utility and Assortative Matching

Matching With Frictions

A simple Shimer and Smith Model

Two-sided Matching with Transferrable Utility

- Less attractive agents may compensate more attractive ones to form a match
- in the labor market: Wage.
- cleaning for roommates, child care in marriage
- We will no focus on **assortative matching**

Assortative Matching

Environment:

- A fixed measure of workers indexed by $x \in \mathbb{X}$ (uniform)
- A fixed measure of jobs indexed by $y \in \mathbb{Y}$ (uniform)
- A production function $f(x, y)$
- Common ranking $f_x > 0, f_y > 0$
- The **cross partial derivatives** of f have a key function for monotone matching.
  - Example 1: $f^+(x, y) = \alpha x^\theta y^\theta$
  - Example 2: $f^-(x, y) = \alpha x^\theta (1 - y)^\theta + g(y)$
- We allow matched agents to transfer each other w (the wage).

Assortative Matching

Preferences:

- Workers care about the wage
- Firm care about profits: $\pi(y) = f(x, y) - w$

Allocation is defined by a matching rule (μ, w):

- $\mu(x) = y$: Which worker matches to which firm. Pure matching.
- $w(x)$: a wage schedule.

Assortative Matching: equlibrium

Stable Matching Rule:

- No pair (x, y) can do better than in equlibrium:

$$\forall x, y : \quad \underbrace{w(x)}_{x \text{ eqm payoff}} + \underbrace{\pi(\mu^{-1}(y), y)}_{y \text{ eqm payoff}} \geq \underbrace{f(x, y)}_{\text{potential output}}$$

Results:

- Existence: Yes. Shapley and Shubik 1971
- Eficiency: Yes. Maximizes joint utility
- Unique: Matching is generically unique, transfers are not
- Stable Matching and Competitive Eqm coincide (Gretsky, Ostroy and Zame 1999)

Competitive Eqm and Assortative Matching

- Firm's problem:
  - Take the wage schedule given and choose x to max profit:

$$\max_x f(x,y) - w(x)$$

  - FOC: $f_x(x,y) - w_x(x) = 0$
  - What is eqm allocation?
  - follows from SOC: $f_{xx}(x,y) - \underbrace{w_{xx}(x)}_{?} < 0$

Competitive Eqm and Assortative Matching

- What's the sign of $w_{xx}(x)$? Take derive of FOC at the Eqm condition $\mu(x) = y$:

$$\frac{d}{dx}\left(f_x(x,\mu(x)) - w_x(x)\right) = 0$$

$$f_{xx}(x,\mu(x)) + f_{xy}(x,\mu(x))\frac{d\mu(x)}{dx} - w_{xx}(x) = 0$$

- so, the SOC is satisfied provided:

$$f_{xx}(x,y) - w_{xx}(x) < 0 \iff$$

$$f_{xy}(x,\mu(x))\frac{d\mu(x)}{dx} > 0$$

- Notice that $f_{xy}(x,\mu(x))\frac{d\mu(x)}{dx}$ measures the **assortative matching relationship**

Production Function and Assortative Matching

- We have:
  1. $\boxed{+ \text{ Assortative Matching (PAM):}}$ $f_{xy}(x, \mu(x)) > 0$ if $\frac{d\mu(x)}{dx} > 0$
  2. $\boxed{- \text{ Assortative Matching (NAM):}}$ $f_{xy}(x, \mu(x)) < 0$ if $\frac{d\mu(x)}{dx} < 0$

Production Function and Assortative Matching

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  2. $\boxed{- \text{ Assortative Matching (NAM):}}$ $f_{xy}(x, \mu(x)) < 0$ if $\frac{d\mu(x)}{dx} < 0$
- f_{xy} describes the $\boxed{\text{supermodularity}}$ of f.
- A function $f : \mathbb{R}^k \to \mathbb{R}$ is **supermodular** if

$$f(x \uparrow y) + f(x \downarrow y) \geq f(x) + f(y)$$

  where $\uparrow, \downarrow$ denote element-wise $\max, \min$ respectively.
- If f is twice differentiable, the condition is equivalent to

$$\frac{\partial^2 f}{\partial z_i \partial z_j} \geq 0, \forall i \neq j.$$

Production Function and Assortative Matching

- We have:
  1. $+$ Assortative Matching (PAM): $f_{xy}(x, \mu(x)) > 0$ if $\frac{d\mu(x)}{dx} > 0$
  2. $-$ Assortative Matching (NAM): $f_{xy}(x, \mu(x)) < 0$ if $\frac{d\mu(x)}{dx} < 0$
- f_{xy} describes the supermodularity of f.
  - if f is super-modular, better workers in better firms is more efficient
  - Gives a clear rationale for why better workers should assortatively match with firms.
- Supermodularity is about the rate of change in the change: Do better workers gain *more* from moving to better firms.
- Note: With pure matching (like here), we cannot differentiate worker from firm effects.

Matching with Frictions: Environment

- A fixed measure of workers indexed by $x \in \mathbb{X}$ (uniform)
- A fixed measure of jobs indexed by $y \in \mathbb{Y}$ (uniform)
- A production function $f(x, y)$
- Common ranking $f_x > 0, f_y > 0$
- We allow matched agents to transfer each other w (the wage).
- unemployed get $b(x)$; vacancies cost $c(y)$
- workers and firms care about EPV (forward looking)

Matching with Frictions: Allocations

- $u(x)$ is the mass of unemployed workers, $v(x)$ is the mass of vacancies
- $h(x, y)$ is the mass of matches (like μ, but not pure anymore!)
- $w(x, y)$ is the wage and $M(x, y)$ the matching decision (yes/no)

Matching Process

- Meeting technology is imperfect:
  - unemployed find offers at rate λ
  - vacancies find workers at rate μ
  - λ and μ can be endogenized with a matching function:
    - the number of matches is $N = m(U, V)$
    - then $\lambda = \frac{N}{U}, \mu = \frac{N}{V}$
    - a classic matching function is $m(u, v) = \alpha u^{0.5} v^{0.5}$
- matching is random: workers draw from $v(y)$, firms draw from $u(x)$

Matching Process: Timing

1. production: matches produce output and pay wage
2. meeting: U and V meet
3. matching: newly matched pairs decide wether to start partnership
4. separation: existing matches destroyed at rate δ

Match Surplus - Present Values

- $W_1(x,y,w)$ and $W_0(x)$ are EPV of employed and unemployed
- $\Pi_1(x,y,w)$ and $\Pi_0(y)$ are EPV of job and vacancy
- Surplus is defined as

$$S(x,y) := W_1(x,y,w) + \Pi_1(x,y,w) - W_0(x) - \Pi_0(y)$$

- Worker EPV: $rW_1(x,y,w) = w + \delta(W_0(x) - W_1(x,y,w))$
- Job EPV: $r\Pi_1(x,y,w) = f(x,y) - w + \delta(\Pi_0(y) - \Pi_1(x,y,w))$

Value of Match Surplus

Some simple algebra gives us that:

$$(r+\delta)S(x,y) = f(x,y) - rW_0(x) - r\Pi_0(y)$$

- Note that we don't need to know the wage to compute this!
- Under TU, the matching decision is $M(x,y) = \mathbf{1}[S(x,y) \geq 0]$
- Surplus can be non-monotonic because of option value!
- Surplus inherits complementarity directly from f.

Wages and Division of Surplus

- There an infinite number of ways to split the surplus
- S-S assume: nash bargaining with α the worker's bargaining power.
- then the optimal wage $w(x,y)$ solves

$$(1-\alpha)\left(W_1(x,y,w)-W_0(y)\right)=\alpha\left(\Pi_1(x,y,w)-\Pi_0(y)\right)$$

Wages and Division of Surplus

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- Therefore, upon meeting
  - worker gets $W_0(x)+\alpha(S(x,y)$
  - firm gets $\Pi_0(x)+(1-\alpha)(S(x,y)$

EPV of unemployed and vacancy

- EPV of the unemployed:

$$rW_0(x) = (1+r)b(x) + \lambda \int \alpha M(x,y)S(x,y)\frac{v(y)}{V}dy$$

- EPV of a vacancy:

$$r\Pi_0(x) = -(1+r)c(y) + \mu \int (1-\alpha)M(x,y)S(x,y)\frac{u(x)}{U}dx$$

- Matching Distribution

$$\delta h(c,y) = \frac{\lambda}{V}M(x,y)u(x)v(y)$$

Equlibrium

Given the primitives $f(x,y), c(y), b(x), r, \delta, \alpha, \lambda, \mu$, a stationary search equilibrium is defined by

- EPVs: $S(x,y), \Pi_0, W_0, \Pi_1, \Pi_0$
- Allocations: $h(x,y), u(x), v(y)$
- wage $w(x,y)$ and matching functions $M(x,y)$

such that

1. the EPVs solve the Bellman Equations
2. the wage is the Nash barginaing solution
3. the distributions satisfy stationarity and adding up propoerties.

Results

- Existence: Yes Shimer and Smith (2000)
- Uniqueness: NO
- Efficiency: Not in general
    - workers do not internalize how the affect others' search (search externality)
    - romm for efficiency improving policies
- Assortative Matching
    - Shimer and Smith (2000) introduce new definitions: monotonicity of matching set boundaries.
    - log supermodular $f(x,y) \to$ PAM
    - log submodular $f(x,y) \to$ NAM
    - this requires stronger complementarities than in frictionless world.

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